

2017

Full Marks : 100

Time : 3 hours

Question 1 carries 20 marks and other questions are of 16 marks each

Answer **six** questions, selecting at least **one** from each Group and Question 1 which is compulsory

1. Choose the correct answers of the following :

(a) The supremum of the set

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}$$

is

- (i) 0
- (ii) 1
- (iii)  $\frac{1}{2}$
- (iv) None of these

(2)

(b) The sequence whose  $n$ th term  $a_n = \frac{1}{n^2}$  is

- (i) convergent and converges to the limit 0
- (ii) divergent and diverges to the limit 1
- (iii) convergent and converges to the limit 1
- (iv) None of these

(c)  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$  is equal to

- (i) 1
- (ii)  $e$
- (iii)  $\frac{1}{e}$
- (iv) None of these

(d) The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if

- (i)  $p < 1$
- (ii)  $p > 1$
- (iii)  $p = 1$
- (iv) None of these

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~~(e)~~ The series

$$\frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \dots \infty$$

is

- (i) convergent
  - (ii) divergent
  - (iii) oscillatory
  - (iv) None of these
- (f) Set of integer with respect to multiplication is
- ~~(i)~~ group
  - (ii) Abelian group
  - (iii) semigroup
  - (iv) None of these
- (g) The order of an element  $w$  of the group  $\{1, w, w^2\}$  is
- ~~(i)~~ 1
  - (ii) 2
  - (iii) 3
  - (iv) 4

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(h) Intersection of two subgroup of a group  $G$  is

- (i) group
  - ~~(ii)~~ subgroup
  - (iii) Abelian group
  - (iv) None of these
- (i) If  $H$  is a nonempty subset of a group  $G$ , then  $H$  is subgroup of  $G$  for every  $a, b \in H$  if

- ~~(i)~~  $ab^{-1} \in H$
  - (ii)  $ab \in H$
  - (iii)  $b/a \in H$
  - (iv) None of these
- (j) Every isomorphic image of a cyclic group is
- (i) group
  - (ii) Abelian group
  - ~~(iii)~~ cyclic group
  - (iv) None of these

(5)

## Group—A

2. (a) Prove that  $\sqrt{2}$  is not a rational number.(b) Find the Dedekind's section corresponding to the sequence  $\{x_n\}$ , where

$$x_n = \sum_{n=1}^{\infty} \frac{1.3.5 \dots (2n+1)}{2.4.6 \dots 2n} \cdot \frac{1}{2^n}$$

3. (a) Prove that a bounded monotonic sequence of rational numbers is convergent.

(b) Show with the help of Cauchy's general principle of convergence that the sequence  $\{a_n\}$ , where

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

is not convergent.

4. (a) Show that the necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

(b) Prove that the sequence  $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \dots$  tends to the positive roots of  $x^2 - x - 7 = 0$  as limit.

(6)

5. (a) State and prove Taylor's theorem with Lagrange's form of remainder. (16)

(b) If

$$f(x) = x^2 \sin \frac{1}{x} \text{ when } x \neq 0$$

$$= 0 \text{ when } x = 0$$

show that  $f(x)$  is continuous and differentiable at  $x = 0$ .

## Group—B

6. (a) State and prove d'Alembert's ratio test.

(b) Test the convergence of the series

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

7. (a) State and prove Cauchy's condensation test. (16)

(b) Test the convergence of the series

$$1 + \frac{x}{2} + \frac{12}{3^2} x^2 + \frac{13}{4^3} x^3 + \dots + \frac{1n}{(n+1)^n} x^n + \dots$$

8. (a) Define absolute convergence. Prove that the terms of an absolutely convergent can be rearranged without affecting the convergence and the sum of the series.

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(Turn Over)

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(Continued)

(7)

(b) If the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

is deranged so that the ratio of the number of positive terms to the number of negative terms tends to  $k > 0$ , show that the sum is  $\frac{1}{2} \log 4k$ .

## Group—C

9. (a) If  $a, b$  be any two elements of a group  $G$ , then prove that  $(ab)^{-1} = b^{-1}a^{-1}$ .

(b) Prove that  $n$ th root of unity is an Abelian group under multiplication.

10. (a) State and prove Lagrange's theorem.

(b) Prove that every subgroup  $H$  of a cyclic group  $G$  is also cyclic group.

11. (a) Prove that every finite integral domain is a field.

(b) Prove that the set of integers is a commutative ring with unit element with respect to usual addition and multiplication.

(8)

12. (a) State and prove fundamental theorem of homomorphism of rings.

(b) Define kernel of a ring homomorphism. Prove that if  $f$  is a homomorphism of a ring into a ring  $R$  with kernel  $S$ , then  $S$  is an ideal of  $R$ .

Handwritten calculations and diagrams on the right page:

- Diagram showing a mapping from set  $A$  to set  $B$  via a function  $f$ . The mapping is defined as  $f(x) = \frac{x}{2}$  for  $x \in A$ . The image of  $A$  under  $f$  is shown as  $B$ .
- Diagram showing a mapping from set  $C$  to set  $D$  via a function  $g$ . The mapping is defined as  $g(y) = \frac{y}{3}$  for  $y \in C$ . The image of  $C$  under  $g$  is shown as  $D$ .
- Diagram showing a mapping from set  $E$  to set  $F$  via a function  $h$ . The mapping is defined as  $h(z) = \frac{z}{4}$  for  $z \in E$ . The image of  $E$  under  $h$  is shown as  $F$ .
- Diagram showing a mapping from set  $G$  to set  $H$  via a function  $i$ . The mapping is defined as  $i(w) = \frac{w}{5}$  for  $w \in G$ . The image of  $G$  under  $i$  is shown as  $H$ .
- Diagram showing a mapping from set  $I$  to set  $J$  via a function  $j$ . The mapping is defined as  $j(v) = \frac{v}{6}$  for  $v \in I$ . The image of  $I$  under  $j$  is shown as  $J$ .
- Diagram showing a mapping from set  $K$  to set  $L$  via a function  $k$ . The mapping is defined as  $k(u) = \frac{u}{7}$  for  $u \in K$ . The image of  $K$  under  $k$  is shown as  $L$ .
- Diagram showing a mapping from set  $M$  to set  $N$  via a function  $l$ . The mapping is defined as  $l(t) = \frac{t}{8}$  for  $t \in M$ . The image of  $M$  under  $l$  is shown as  $N$ .
- Diagram showing a mapping from set  $O$  to set  $P$  via a function  $m$ . The mapping is defined as  $m(s) = \frac{s}{9}$  for  $s \in O$ . The image of  $O$  under  $m$  is shown as  $P$ .
- Diagram showing a mapping from set  $Q$  to set  $R$  via a function  $n$ . The mapping is defined as  $n(r) = \frac{r}{10}$  for  $r \in Q$ . The image of  $Q$  under  $n$  is shown as  $R$ .
- Diagram showing a mapping from set  $S$  to set  $T$  via a function  $o$ . The mapping is defined as  $o(p) = \frac{p}{11}$  for  $p \in S$ . The image of  $S$  under  $o$  is shown as  $T$ .
- Diagram showing a mapping from set  $U$  to set  $V$  via a function  $p$ . The mapping is defined as  $p(q) = \frac{q}{12}$  for  $q \in U$ . The image of  $U$  under  $p$  is shown as  $V$ .
- Diagram showing a mapping from set  $W$  to set  $X$  via a function  $q$ . The mapping is defined as  $q(r) = \frac{r}{13}$  for  $r \in W$ . The image of  $W$  under  $q$  is shown as  $X$ .
- Diagram showing a mapping from set  $Y$  to set  $Z$  via a function  $r$ . The mapping is defined as  $r(s) = \frac{s}{14}$  for  $s \in Y$ . The image of  $Y$  under  $r$  is shown as  $Z$ .
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