2017

Full Marks : 100

Time : 3 hours

Question **1** *carries 20 marks and other questions are of 16 marks each*

Answer **six** questions, selecting at least **one** from each Group and Question **1** which is compulsory

1. Choose the correct answers of the following :

(a) The supremum of the set

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$$

is

(i) 0

(ii) 1

(iii) $\frac{1}{2}$

(iv) None of these

(Turn Over)

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(b) The sequence whose *n*th term $a_n = \frac{1}{n^2}$ is

(i) convergent and converges to the limit 0

(ii) divergent and diverges to the limit 1

(iii) convergent and converges to the limit 1

(iv) None of these

(c) $\underset{n \to \infty}{dt} \left(1 + \frac{1}{n}\right)^n$ is equal to

(i) 1

(ii) e

(iii) $\frac{1}{e}$

(iv) None of these

(d) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if

(i) $p < 1$

(ii) $p > 1$

(iii) $p = 1$

(iv) None of these

(Continued)

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(e) The series

$$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\cdots\infty$$

is

(i) convergent

(ii) divergent

(iii) oscillatory

(iv) None of these

(f) Set of integer with respect to multiplication is

(i) group

(ii) Abelian group

(iii) semigroup

(iv) None of these

(g) The order of an element w of the group $\{1, w, w^2\}$ is

(i) 1

(ii) 2

(iii) 3

(iv) 4

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(h) Intersection of two subgroup of a group G is

(i) group

(ii) subgroup

(iii) Abelian group

(iv) None of these

(i) If H is a nonempty subset of a group G, then H is subgroup of G for every $a, b \in H$ if

(i) $ab^{-1} \in H$

(ii) $ab \in H$

(iii) $b/a \in H$

(iv) None of these

(j) Every isomorphic image of a cyclic group is

(i) group

(ii) Abelian group

(iii) cyclic group

(iv) None of these

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Group—A

2. (a) Prove that $\sqrt{2}$ is not a rational number.

(b) Find the Dedekind's section corresponding to the sequence $\langle x_n \rangle$, where

$$x_n = \sum_{n=1}^{n} \frac{1.3.5.\cdots(2n+1)}{2.4.6.\cdots 2n} \cdot \frac{1}{2^n}$$

3. (a) Prove that a bounded monotonic sequence of rational numbers is convergent.

(b) Show with the help of Cauchy's general principle of convergence that the sequence $\langle a_n \rangle$, where

$$a_n = \sum_{n=1}^{n} \frac{1}{n}$$

is not convergent.

4. (a) Show that the necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

(b) Prove that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \cdots$ tends to the positive roots of $x^2 - x - 7 = 0$ as limit.

33

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5. (a) State and prove Taylor's theorem with Lagrange's form of remainder. 16

(b) If

$$f(x) = x^2 \sin\frac{1}{x} \quad \text{when } x \neq 0$$
$$= 0 \quad \text{when } x = 0$$

show that $f(x)$ is continuous and differentiable at $x = 0$.

Group—B

6. (a) State and prove d'Alembert's ratio test.

(b) Test the convergence of the series

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \cdots$$

7. (a) State and prove Cauchy's condensation test. 16

(b) Test the convergence of the series

$$1 + \frac{x}{2} + \frac{\lfloor 2}{3^2} x^2 + \frac{\lfloor 3}{4^3} x^3 + \cdots + \frac{\lfloor n}{(n+1)^n} x^n + \cdots$$

8. (a) Define absolute convergence. Prove that the terms of an absolutely convergent can be rearranged without affecting the convergence and the sum of the series.

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(b) If the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

is deranged so that the ratio of the number of positive terms to the number of negative terms tends to $k>0$, show that the sum is $\frac{1}{2}\log 4k$.

Group—C

9. (a) If a, b be any two elements of a group G, then prove that $(ab)^{-1}=b^{-1}a^{-1}$.

   (b) Prove that nth root of unity is an Abelian group under multiplication.

10. (a) State and prove Lagrange's theorem.

    (b) Prove that every subgroup H of a cyclic group G is also cyclic group.

11. (a) Prove that every finite integral domain is a field.

    (b) Prove that the set of integers is a commutative ring with unit element with respect to usual addition and multiplication.

12. (a) State and prove fundamental theorem of homomorphism of rings.

    (b) Define kernel of a ring homomorphism. Prove that if f is a homomorphism of a ring into a ring R with kernel S, then S is an ideal of R.

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